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Poisson variations in the problem of the stability of equilibria in rigid body mechanics☆

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ABSTRACT

The existence and stability of equilibria in rigid body mechanics is considered. A class of variations is indicated which satisfy the analogue of Poisson's equations, suitable for use when investigating both the sufficient and necessary conditions for the stability of such equilibria and which, in particular, enable the invariance of the equations of motion of the system and their first integrals to be effectively used when the phase variables and parameters of the problem are interchanged. The result is illustrated using the example of the problem of the motion of a gyrostat far from attracting objects.

As is well known[1] (see also, for example, Ref. 2), the steady motions of mechanical systems with first integrals can be sought as critical points of one of these integrals, considered as a function on a joint fixed level of the remaining integrals. For this purpose Routh's function is used, namely, a linear combination composed of this integral and the remaining integrals, multiplied by Lagrange multipliers. The sufficient conditions for the stability of the steady motions obtained are satisfied if the restriction of the second variation of Routh's function on a linear manifold, defined by the joint level of the remaining integrals, is a sign-definite quadratic form.

The investigation of the sign-definiteness of the restriction of quadratic forms on a linear manifold goes back, probably, to Weierstrass (see, for example, Ref. 3). As was found from solving specific problems, the direct method, based on the expression from linear constraints of some of the variables as a function of the remaining ones and subsequent substitution of the expressions obtained into the quadratic form being investigated, encounters purely algebraic difficulties connected with the unwieldy nature of the expressions obtained and, often, with losses of their symmetrical form. Efforts have been made to find alternative ways of solving the problem. Such a solution, based on an investigation of the determinant of a so-called extended matrix, was given, in particular, by Mann.[4] Later the criterion he obtained has been rediscovered many times (see, for example, Ref. 5; for a brief proof see Ref. 6).

The problem of the sign-definiteness of the constraints of quadratic forms, actively employed by economists to solve problems of non-linear programming,[7] attracted the attention of mathematicians in view of the development by Chetayev[8] of Routh's theory, which enabled the existence and stability of the steady motions of mechanical systems with first integrals to be investigated. This sign-definiteness criterion was not only reestablished but also applied to the problem of the motion of a gyrostat in a circular orbit in a central force field.[9] It turned out that invariance of the functions that arise in the problem occurs with respect to cyclic permutations of the indices both in the space of the variables and in the space of the parameters. This property has been used considerably in investigations for providing an answer in symmetrical form. The criterion of sign-definiteness in a form[10] which takes this symmetry into account, was proved and employed in the problem of the motion of satellite-gyroscope with tethered elements.

Below we develope the idea, based on "direct" elimination of dependent variations and resting on the property of a group of rotations, which define the configuration space in the problem of the motion of a rigid body. In this case the symmetry with respect to cyclic permutations of the indices must be borne in mind, knowing the particular features of the kinematic equations of motion of a rigid body. A similar idea, based on the use of discrete symmetry in the mass distribution of a rigid body was developed earlier in Ref. 11.

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1. Formulation of the problem

Equations of motion. First integrals. In a number of problems of rigid-body mechanics, the equations of motion are represented in the form of a system of "Euler–Poisson equations"

$$\frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{\omega}} = \frac{\partial L}{\partial \boldsymbol{\omega}} \times \boldsymbol{\omega} + \mathbf{Q}; \quad L = L(\boldsymbol{\omega}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}), \quad \mathbf{Q} = \frac{\partial L}{\partial \boldsymbol{\alpha}} \times \boldsymbol{\alpha} + \frac{\partial L}{\partial \boldsymbol{\beta}} \times \boldsymbol{\beta} + \frac{\partial L}{\partial \boldsymbol{\gamma}} \times \boldsymbol{\gamma} \tag{1.1}$$

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \omega, \quad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega} \tag{1.2}$$

where L is the Lagrange function and **Q** is the moment of the forces.

The unit vectors, orthogonal to one another,

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)^T, \quad \boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)^T, \quad \boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)^T$$

form a right triplet of the system of coordinates $OX_\alpha X_\beta X_\gamma$, relative to which the motion is considered. The components of these vectors in a system of coordinates $Ox_1x_2x_3$, rigidly connected with the body, form the orthogonal matrix

$$\mathbf{S} = \begin{Vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{Vmatrix} \tag{1.3}$$

It gives the position of the body with respect to the system of coordinates $OX_\alpha X_\beta X_\gamma$.

The components of the matrix **S** change with time during the motion. The vector of the angular velocity $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T$ is then determined by the components of the matrix

$$\hat{\boldsymbol{\omega}} = \mathbf{S}^{-1} \cdot \dot{\mathbf{S}} \tag{1.4}$$

which, in view of the orthogonality of the matrix **S**, is skew symmetric

$$\hat{\boldsymbol{\omega}} = \begin{Vmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{Vmatrix} \tag{1.5}$$

Equality (1.4) can be represented in the form

$$\dot{\mathbf{S}} = \mathbf{S} \cdot \hat{\boldsymbol{\omega}} \tag{1.6}$$

and defines the matrix form of Poisson's equations (1.2).

By virtue of Poisson's equations (1.2) there are two triplets of geometric integrals

$$\mathcal{T}_\alpha = (\boldsymbol{\alpha}, \boldsymbol{\alpha}) - 1 = 0, \quad (\boldsymbol{\alpha\beta\gamma}) \tag{1.7}$$

$$\mathcal{T}_{\alpha\beta} = (\boldsymbol{\alpha}, \boldsymbol{\beta}) = 0, \quad (\boldsymbol{\alpha\beta\gamma}) \tag{1.8}$$

In each of these two groups the first integrals are obtained from each other by cyclic (α, β, γ) permutation. Relations (1.7) and (1.8) express the orthonormality of the system of vectors α, β and γ. These integrals can be represented in matrix form as follows:

$$\mathbf{S} \cdot \mathbf{S}^T = \mathbf{E} \tag{1.9}$$

where **E** is the identity 3×3 matrix.

Since the Lagrange function L is explicitly independent of time, we have the generalized energy integral (the Painlevé–Jacobi integral)

$$J_0 = \left(\frac{\partial L}{\partial \boldsymbol{\omega}}, \boldsymbol{\omega}\right) - L \tag{1.10}$$

Remark 1. It is said that Eqs. (1.1) and (1.2) define the dynamics on the group of rotation SO(3).

2. The geometry of the configuration and phase spaces

The variables α, β and γ, which define the position of the system, are dependent – the joint level of the first integrals (1.7) and (1.8) defines the three-dimensional configuration space $\mathbf{M}^3$.

We will consider the one-parameter family of positions of the system, defined by the orthogonal matrices

$$\mathbf{S}_\varepsilon = \mathbf{S} + \varepsilon\delta\mathbf{S} + \dots \tag{2.1}$$

which depend on the parameter $\varepsilon \in R$. Since the orthogonality condition

$$\mathbf{S}_\varepsilon \cdot \mathbf{S}_\varepsilon^T = \mathbf{E}$$

is satisfied identically with respect to ε, the components of the matrix $\delta\mathbf{S}$ satisfy the matrix equation

$$\mathbf{S} \cdot \delta\mathbf{S}^T + \delta\mathbf{S} \cdot \mathbf{S}^T = 0 \tag{2.2}$$

which can be represented in vector form as follows:

$$\delta\mathcal{T}_{\alpha} = (\boldsymbol{\alpha}, \delta\boldsymbol{\alpha}) = 0 \quad (\boldsymbol{\alpha\beta\gamma}) \tag{2.3}$$

In other words, if $\mathbf{M}^3$ is a three-

$$\delta\mathcal{T}_{\alpha\beta} = (\boldsymbol{\alpha}, \delta\boldsymbol{\beta}) + (\boldsymbol{\beta}, \delta\boldsymbol{\alpha}) = 0 \quad (\boldsymbol{\alpha\beta\gamma}) \tag{2.4}$$

In other words, if $\mathbf{M}^3$ is a three-dimensional configuration space, defined by relations (1.7) and (1.8) or by relation (2.1), the tangential space $\mathbf{T_x M}^3$ at the point $\mathbf{x} \in \mathbf{M}^3$ is given by relations (2.3) and (2.4) or by relation (2.2).

Basic assertion. The independent coordinates $\delta\theta = (\delta\theta_1, \delta\theta_2, \delta\theta_3)^T$ in the tangential space $\mathbf{T_x M}^3$ can be determined using Poisson's equations, represented in the form

$$\delta\boldsymbol{\alpha} = \boldsymbol{\alpha} \times \delta\boldsymbol{\theta}, \quad \delta\boldsymbol{\beta} = \boldsymbol{\beta} \times \delta\boldsymbol{\theta}, \quad \delta\boldsymbol{\gamma} = \boldsymbol{\gamma} \times \delta\boldsymbol{\theta} \tag{2.5}$$

or, in matrix form,

$$\delta\mathbf{S} = \mathbf{S} \cdot \delta\hat{\boldsymbol{\theta}}; \quad \delta\hat{\boldsymbol{\theta}} = \begin{Vmatrix} 0 & -\delta\theta_3 & \delta\theta_2 \\ \delta\theta_3 & 0 & -\delta\theta_1 \\ -\delta\theta_2 & \delta\theta_1 & 0 \end{Vmatrix} \tag{2.6}$$

The proof of the matrix version of the assertion, which consists of substituting expression (2.6) into (2.2), rests on the property of skew symmetry of the matrix $\delta\hat{\theta}$. The proof of the vector version of the assertion rests on the properties of the mixed product and reduces to substituting expressions (2.5) into relations (2.3) and (2.4).

3. Equilibria and the sufficient conditions for their stability

The existence of steady motions. According Routh's theory,[1,2] for the first integrals indicated, steady motions of the system are defined as critical points of Routh's function

$$W = \mathcal{T}_0 + \frac{1}{2}\sum \lambda_{\alpha}\mathcal{T}_{\alpha} + \sum \lambda_{\alpha\beta}\mathcal{T}_{\alpha\beta} \tag{3.1}$$

Here and henceforth, unless otherwise stated, the summation is carried out over cyclic permutation of the indices (α,β,γ).

These critical points are found from the equations

$$\frac{\partial W}{\partial \boldsymbol{\omega}} = \frac{\partial^2 L}{\partial \boldsymbol{\omega}^2}\boldsymbol{\omega} + \frac{\partial L}{\partial \boldsymbol{\omega}} - \frac{\partial L}{\partial \boldsymbol{\omega}} = \frac{\partial^2 L}{\partial \boldsymbol{\omega}^2}\boldsymbol{\omega} = 0$$

which allow of the unique solutions

$$\boldsymbol{\omega} = 0 \tag{3.2}$$

when the Lagrange function is non-degenerate with respect to the angular velocities. This indicates that these steady motions are equilibria.

The equilibrium configurations themselves are found from the three systems of equations

$$\frac{\partial W}{\partial \boldsymbol{\alpha}} = \frac{\partial^2 L}{\partial \boldsymbol{\alpha} \partial \boldsymbol{\omega}}\boldsymbol{\omega} - \frac{\partial L}{\partial \boldsymbol{\alpha}} + \lambda_{\alpha}\boldsymbol{\alpha} + \lambda_{\alpha\beta}\boldsymbol{\beta} + \lambda_{\gamma\alpha}\boldsymbol{\gamma} = 0 \quad (\boldsymbol{\alpha\beta\gamma}) \tag{3.3}$$

which are obtained from one another by cyclic permutation of the indices, indicated above. These three systems, by virtue of Eq. (3.2), can be written in the form

$$\frac{\partial U}{\partial \boldsymbol{\alpha}} + \lambda_{\alpha}\boldsymbol{\alpha} + \lambda_{\alpha\beta}\boldsymbol{\beta} + \lambda_{\gamma\alpha}\boldsymbol{\gamma} = 0 \quad (\boldsymbol{\alpha\beta\gamma}) \quad U = U(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) = -L(0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) \tag{3.4}$$

The function U is called a potential.

Hence, the equilibria are found from the nine equations (3.4) and the six equations (1.7) and (1.8) in terms of nine unknown direction cosines α, β and γ and six unknown Lagrange multipliers

$$\boldsymbol{\lambda} = (\lambda_{\alpha}, \lambda_{\beta}, \lambda_{\gamma}, \lambda_{\alpha\beta}, \lambda_{\beta\gamma}, \lambda_{\gamma\alpha})^T$$

By virtue of system (3.4), and also Eqs. (1.7) and (1.8)

$$\lambda_{\alpha} = -\left(\frac{\partial U}{\partial \boldsymbol{\alpha}}, \boldsymbol{\alpha}\right), \quad \lambda_{\alpha\beta} = -\left(\frac{\partial U}{\partial \boldsymbol{\alpha}}, \boldsymbol{\beta}\right) = -\left(\frac{\partial U}{\partial \boldsymbol{\beta}}, \boldsymbol{\alpha}\right) \quad (\boldsymbol{\alpha\beta\gamma}) \tag{3.5}$$

Remark 2. At first glance it would appear that, instead of solving the unwieldy problem of investigating the properties of a conditional extremum, the solution of which is found from Eqs. (3.4), (1.7) and (1.8), it would be more convenient to introduce some angles, which define the orientation of the body and thereby confine ourselves to solving the problem of finding extremum points of a trigonometric function of three variables and investigating their types. Undoubtedly, the use of modern computer algebra software, such as MATHEMATICA or MAPLE, enables both the necessary and sufficient conditions for the stability of already known solutions to be written quite easily in a certain form. Nevertheless, the analytical solution which arises, that depends on the parameters of trigonometric equations, and also the representation of the stability conditions in a symmetrical form convenient for investigation, sometimes turns out to be more difficult for these techniques and their user than finding some other solutions of system (3.4), (1.7), (1.8), which, as a rule, is algebraic. When solving it, also as a rule, one must effectively employ some symmetry of the problem, for example, as occurs in the example considered below.

Remark 3. System (3.4), (1.7), (1.8) can be interpreted as the condition for the 1-form to be equal to zero

$$\delta W = \sum\left[\frac{\partial W}{\partial \boldsymbol{\alpha}}\delta\boldsymbol{\alpha} + \frac{\partial W}{\partial \lambda_{\alpha}}\delta\lambda_{\alpha} + \frac{\partial W}{\partial \lambda_{\alpha\beta}}\delta\lambda_{\alpha\beta}\right] = 0$$

By substituting expressions (2.5) into this relation and equating the independent variations to zero we obtain the equilibrium condition in the form

$$\frac{\partial U}{\partial \boldsymbol{\alpha}} \times \boldsymbol{\alpha} + \frac{\partial U}{\partial \boldsymbol{\beta}} \times \boldsymbol{\beta} + \frac{\partial U}{\partial \boldsymbol{\gamma}} \times \boldsymbol{\gamma} = 0 \tag{3.6}$$

and also six geometric integrals. Relation (3.6) expresses the fact that the moment of the forces **Q** are equal to zero.

The sufficient conditions for the stability of the equilibria. To investigate the sufficient conditions for stability, according to Routh's theorem, we write out the second variation of Routh's function on the equilibria obtained

$$2\delta^2 W = \left(\frac{\partial^2 \mathcal{T}_0}{\partial \boldsymbol{\omega}^2}\delta\boldsymbol{\omega}, \delta\boldsymbol{\omega}\right) + 2\sum\left(\frac{\partial^2 \mathcal{T}_0}{\partial \boldsymbol{\alpha}\partial\boldsymbol{\omega}}\delta\boldsymbol{\omega}, \delta\boldsymbol{\alpha}\right) + \sum\left(\frac{\partial^2 \mathcal{T}_0}{\partial \boldsymbol{\alpha}^2}\delta\boldsymbol{\alpha}, \delta\boldsymbol{\alpha}\right) +$$
$$+ 2\sum\left(\frac{\partial^2 \mathcal{T}_0}{\partial \boldsymbol{\alpha}\partial\boldsymbol{\beta}}\delta\boldsymbol{\beta}, \delta\boldsymbol{\alpha}\right) + \sum\lambda_{\alpha}(\delta\boldsymbol{\alpha}, \delta\boldsymbol{\alpha}) + 2\sum\lambda_{\alpha\beta}(\delta\boldsymbol{\alpha}, \delta\boldsymbol{\beta})$$

and investigate the sign-definiteness of its restriction on a linear manifold, defined by relations (2.3) and (2.4). Since, in this solution

$$\frac{\partial^2 J_0}{\partial \boldsymbol{\omega}^2} = \frac{\partial^2 L}{\partial \boldsymbol{\omega}^2}$$

$$\frac{\partial^2 \mathcal{T}_0}{\partial \boldsymbol{\alpha}\partial\boldsymbol{\omega}} = 0, \quad \frac{\partial^2 \mathcal{T}_0}{\partial \boldsymbol{\alpha}^2} = \frac{\partial^2 U}{\partial \boldsymbol{\alpha}^2}, \quad \frac{\partial^2 \mathcal{T}_0}{\partial \boldsymbol{\alpha}\partial\boldsymbol{\beta}} = \frac{\partial^2 U}{\partial \boldsymbol{\alpha}\partial\boldsymbol{\beta}} \quad (\boldsymbol{\alpha\beta\gamma})$$

which can be verified by a direct check, in view of the fact that the quadratic form

$$\delta^2_{\omega} W = \left(\frac{\partial^2 L}{\partial \boldsymbol{\omega}^2}\delta\boldsymbol{\omega}, \delta\boldsymbol{\omega}\right)$$

defined by the kinetic energy of the mechanical system, is positive definite, the problem reduces to investigating the possibility that the restriction of the quadratic form

$$2\delta^2_{\lambda} U = \sum\left(\left(\frac{\partial^2 U}{\partial \boldsymbol{\alpha}^2} + \lambda_{\alpha}\mathbf{E}\right)\delta\boldsymbol{\alpha}, \delta\boldsymbol{\alpha}\right) + 2\sum\left(\left(\frac{\partial^2 U}{\partial \boldsymbol{\alpha}\partial\boldsymbol{\beta}} + \lambda_{\alpha\beta}\mathbf{E}\right)\delta\boldsymbol{\alpha}, \delta\boldsymbol{\beta}\right)$$

on the linear manifold (2.3), (2.4) is sign-definite.

The use of the representation of the linear manifold in the form (2.5) enables us to represent this restriction in the form of a quadratic form of three variables, which have the form

$$2\delta^2_{\theta} W = (\boldsymbol{\Theta}\delta\boldsymbol{\theta}, \delta\boldsymbol{\theta}) = \sum\left(\left(\frac{\partial^2 U}{\partial \boldsymbol{\alpha}^2} + \lambda_{\alpha}\mathbf{E}\right)(\boldsymbol{\alpha} \times \delta\boldsymbol{\theta}), \boldsymbol{\alpha} \times \delta\boldsymbol{\theta}\right) +$$
$$+ 2\sum\left(\left(\frac{\partial^2 U}{\partial \boldsymbol{\alpha}\partial\boldsymbol{\beta}} + \lambda_{\alpha\beta}\mathbf{E}\right)(\boldsymbol{\beta} \times \delta\boldsymbol{\theta}), \boldsymbol{\alpha} \times \delta\boldsymbol{\theta}\right) \tag{3.7}$$

The expressions for the coefficients of the second variations are fairly lengthy, and we will not write them in general form. However, its investigation often turns out to be simpler than an investigation of the properties of the fifteenth-order determinant that arises in the general case when using Mann's criterion[4] or its symmetrisation.[10] We will show this using a mechanical example.

4. Example

The motion of a gyrostat far from attracting centres. "Stellar Watch". We will consider the classical problem of the motion of a gyrostat of mass m attracted by distant attracting objects, assumed fixed in absolute space. Suppose C is the centre of mass of the body, $Cx_1x_2x_3$ is a moving system of coordinates connected with the housing of the gyrostat, the axis of which is directed along the principal central axes of inertia of the gyrostat, $\mathbf{v} = (\upsilon_1, \upsilon_2, \upsilon_3)^T$ is the velocity of the centre of mass of the body and $\mathbf{r} = (r_1, r_2, r_3)^T$ is the vector OC, which specifies the position of the centre of mass.

The kinetic energy of the body is

$$T = \frac{1}{2}[(\mathbf{I}\boldsymbol{\omega}, \boldsymbol{\omega}) + m\mathbf{v}^2] + (\mathbf{K}, \boldsymbol{\omega}), \quad \mathbf{I} = \mathrm{diag}(I_1, I_2, I_3), \quad \mathbf{K} = (K_1, K_2, K_3)^T \tag{4.1}$$

where **I** is the inertial tensor of the body and **K** is the constant gyrostatic momentum vector.

The potential energy of the Newtonian attraction forces, assuming the body to be small compared with the distance to the attracting objects, can be written as

$$U = U_0 + U_1 + U_2 + \ldots \quad U_0 = -mf\sum M_i/r_i, \quad U_1 = 0 \tag{4.2}$$

where M_i is the mass of the i-th attracting object and r_i is its distance to the point C. Here summation is carried out over all the attracting objects. The function U_2, for a certain choice of the absolute system of coordinates $OX_\alpha X_\beta X_\gamma$, takes the form (see, for example, Ref. 12)

$$U(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) = \frac{\varkappa_\alpha}{2}(\mathbf{I}\boldsymbol{\alpha},\boldsymbol{\alpha}) + \frac{\varkappa_\beta}{2}(\mathbf{I}\boldsymbol{\beta},\boldsymbol{\beta}) + \frac{\varkappa_g}{2}(\mathbf{I}\boldsymbol{\gamma},\boldsymbol{\gamma}) \tag{4.3}$$

where the coefficients χ_α, χ_β, χ_γ, depend on the distances to the attracting objects.

We will assume that terms higher than the third power in the expansion of the potential are negligibly small compared with U_0 and U_2. The term U_0 in the expansion of the potential then defines the motion of its centre of mass, and the motion of the gyrostat around the centre of mass does not affect this motion.[13,14]

At the same time, the motion of the gyrostat around the centre of mass is governed both by the term U_2 in the expansion of the potential, and by the motion of the centre of mass itself. For simplicity we will assume that the centre of mass is in an equilibrium position with respect to absolute space. This can be, for example, a libration point or any other point if it is assumed that additional forces act on the gyrostat which guarantee that its centre of mass is at rest and do not produce a moment about the centre of mass. Then $\nu = 0$ and the equations of motion of the gyrostat around the centre of mass can be written in the form

$$\frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{\omega}} = \frac{\partial L}{\partial \boldsymbol{\omega}} \times \boldsymbol{\omega} + \frac{\partial L}{\partial \boldsymbol{\alpha}} \times \boldsymbol{\alpha} + \frac{\partial L}{\partial \boldsymbol{\beta}} \times \boldsymbol{\beta} + \frac{\partial L}{\partial \boldsymbol{\gamma}} \times \boldsymbol{\gamma}; \quad L = T - U_2$$

which must be considered together with Poisson's equations (1.2).

The equilibria of the body, by Routh's method, are defined as critical points of the potential U_2, considered as a function on a joint level of geometric first integrals. In the case when the moments of inertia of the body are unequal, the problem has twenty four solutions of the same kind, in which the axes of the body are elongated along the axes of the absolute system of coordinates. One of these solutions is

$$\begin{aligned} &\alpha_1 = 1, \quad \alpha_2 = 0, \quad \alpha_3 = 0 \\ &\beta_1 = 0, \quad \beta_2 = 1, \quad \beta_3 = 0 \\ &\gamma_1 = 0, \quad \gamma_2 = 0, \quad \gamma_3 = 1 \end{aligned}$$

For this solution

$$\lambda_\alpha = -\varkappa_\alpha I_1, \quad \lambda_\beta = -\varkappa_\beta I_2, \quad \lambda_\gamma = -\varkappa_\gamma I_3, \quad \lambda_{\alpha\beta} = \lambda_{\beta\gamma} = \lambda_{\gamma\alpha} = 0$$

and the linear manifold is defined by the relations

$$\begin{aligned} &\delta\alpha_1 = 0, \quad \delta\alpha_2 = -\delta\theta_3, \quad \delta\alpha_3 = \delta\theta_2 \\ &\delta\beta_1 = -\delta\theta_3, \quad \delta\beta_2 = 0, \quad \delta\beta_3 = \delta\theta_1 \\ &\delta\gamma_1 = \delta\theta_2, \quad \delta\gamma_2 = -\delta\theta_1, \quad \delta\gamma_3 = 0 \end{aligned}$$

The second variation, after reduction, has the form

$$\begin{aligned} \delta^2 W &= \sum((\varkappa_\alpha \mathbf{I} + \lambda_\alpha \mathbf{E})\delta\boldsymbol{\alpha}, \delta\boldsymbol{\alpha}) = (\varkappa_\beta - \varkappa_\gamma)(I_3 - I_2)\delta^2\theta_1 + \\ &+ (\varkappa_\gamma - \varkappa_\alpha)(I_1 - I_3)\delta^2\theta_2 + (\varkappa_\alpha - \varkappa_\beta)(I_2 - I_1)\delta^2\theta_3 \end{aligned} \tag{4.4}$$

Hence, if all three coefficients of the quadratic form (4.4) are positive, the degree of instability of the solution is equal to zero, and secular stability occurs. If the number of negative coefficients is equal to one or three, the solution is unstable;[2] if the number of negative coefficients is equal to two, stability in the first approximation, i.e., gyroscopic stabilization, is possible.

Suppose, to be specific,

$$\varkappa_\alpha < \varkappa_\beta < \varkappa_\gamma \tag{4.5}$$

Then the conditions for the quadratic form (4.4) to be positive definite take the form

$$I_3 < I_2 < I_1 \tag{4.6}$$

In other words, in stable equilibrium the body is stretched along the field components with the greatest intensity and compressed along the field components with the least intensity.

Remark 4. The presence of rotors has no effect of the existence of equilibrium configurations, but it does affect the necessary conditions for them to be stable.

The necessary stability conditions. To investigate the necessary stability conditions we will write equations in variations which, in this case, for solutions determined from relations (3.2) and (3.4), have the form

$$\frac{d}{dt}\delta\left(\frac{\partial L}{\partial \boldsymbol{\omega}}\right) = \frac{\partial L}{\partial \boldsymbol{\omega}} \times \delta\boldsymbol{\omega} + \delta\mathbf{Q}$$
$$\delta\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \delta\boldsymbol{\omega}, \quad \delta\dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \delta\boldsymbol{\omega}, \quad \delta\dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \delta\boldsymbol{\omega} \tag{4.7}$$

$$\delta\left(\frac{\partial L}{\partial \boldsymbol{\omega}}\right) = \frac{\partial^2 L}{\partial \boldsymbol{\omega}^2}\delta\boldsymbol{\omega} + \frac{\partial^2 L}{\partial \boldsymbol{\omega}\partial \boldsymbol{\alpha}}\delta\boldsymbol{\alpha} + \frac{\partial^2 L}{\partial \boldsymbol{\omega}\partial \boldsymbol{\beta}}\delta\boldsymbol{\beta} + \frac{\partial^2 L}{\partial \boldsymbol{\omega}\partial \boldsymbol{\gamma}}\delta\boldsymbol{\gamma}$$
$$\delta\mathbf{Q} = \frac{\partial \mathbf{Q}}{\partial \boldsymbol{\omega}}\delta\boldsymbol{\omega} + \frac{\partial \mathbf{Q}}{\partial \boldsymbol{\alpha}}\delta\boldsymbol{\alpha} + \frac{\partial \mathbf{Q}}{\partial \boldsymbol{\beta}}\delta\boldsymbol{\beta} + \frac{\partial \mathbf{Q}}{\partial \boldsymbol{\gamma}}\delta\boldsymbol{\gamma} \tag{4.8}$$

Hence, there are twelve first-order equations, and the eigenfrequency problem consists of investigating the roots of a twelve-degree polynomial, many of which, however, are equal to zero.

We will use the coordinates in tangential space, introduced above, to reduce the order of this system. To do this we recall the idea of a trajectory variation, which was tacitly used when deriving Eqs. (4.7) and (4.8). We will assume that, in the space $\mathbf{M}^3$, together with the given parametrized time t of the curve $\mathbf{S}(t)$, there is a family of variations of this curve, one-parametric with respect to ε

$$\mathbf{S}_\varepsilon(t) = \mathbf{S}(t) + \varepsilon\delta\mathbf{S}(t) + \ldots$$

Retaining the subscript ε for all the attributes of the variation of the curve, we calculate the derivatives of the left- and right-hand sides of this equation with respect to t and with respect to ε for $\varepsilon = 0$. Since

$$\dot{\mathbf{S}}_\varepsilon = \mathbf{S}_\varepsilon \cdot \hat{\omega}_\varepsilon, \quad \frac{\partial \mathbf{S}_\varepsilon}{\partial \varepsilon} = \mathbf{S}_\varepsilon \cdot \delta\hat{\boldsymbol{\theta}}_\varepsilon$$

we have, on one side,

$$\left(\frac{\partial^2 \mathbf{S}}{\partial\varepsilon\partial t}\right)_{\varepsilon=0} = \left(\frac{\partial}{\partial\varepsilon}[\mathbf{S}_\varepsilon \cdot \hat{\boldsymbol{\omega}}_\varepsilon]\right)_{\varepsilon=0} = \left(\frac{\partial \mathbf{S}_\varepsilon}{\partial\varepsilon} \cdot \hat{\boldsymbol{\omega}}_\varepsilon + \mathbf{S}_\varepsilon \cdot \frac{\partial\hat{\boldsymbol{\omega}}_\varepsilon}{\partial\varepsilon}\right)_{\varepsilon=0} = \delta\mathbf{S} \cdot \hat{\boldsymbol{\omega}} + \mathbf{S} \cdot \delta\hat{\boldsymbol{\omega}} \tag{4.9}$$

and on the other side,

$$\left(\frac{\partial^2 \mathbf{S}}{\partial t\partial\varepsilon}\right)_{\varepsilon=0} = \left(\frac{\partial}{\partial t}\left[\mathbf{S}_\varepsilon \cdot \delta\hat{\boldsymbol{\theta}}_\varepsilon\right]\right)_{\varepsilon=0} = \left(\frac{\partial \mathbf{S}_\varepsilon}{\partial t} \cdot \delta\hat{\boldsymbol{\theta}}_\varepsilon + \mathbf{S}_\varepsilon \cdot \frac{\partial\delta\hat{\boldsymbol{\theta}}_\varepsilon}{\partial t}\right)_{\varepsilon=0} = \dot{\mathbf{S}} \cdot \delta\hat{\boldsymbol{\theta}} + \mathbf{S} \cdot \frac{\partial}{\partial t}\delta\hat{\boldsymbol{\theta}} \tag{4.10}$$

The left-hand sides of relations (4.9) and (4.10) are identical. Equating their right-hand sides, multiplying the equality obtained on the left by $\mathbf{S}^{-1}$ and using equalities (1.6) and (2.6), we obtain after reduction

$$\frac{d}{dt}\delta\hat{\boldsymbol{\theta}} = \delta\hat{\boldsymbol{\omega}} + \delta\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\omega}} - \hat{\boldsymbol{\omega}} \cdot \delta\hat{\boldsymbol{\theta}}$$

or, in vector notation,

$$\delta\dot{\boldsymbol{\theta}} = \delta\boldsymbol{\omega} + \delta\boldsymbol{\theta} \times \boldsymbol{\omega} \tag{4.11}$$

"Euler's kinematic equations" (4.11), which give the rate of change of the variation of the vector $\delta\theta$, are well known. They are used, for example, when deriving Euler's equations, describing the motion of a rigid body about a fixed point, from Hamilton's variational principle.

In view of the fact that the angular velocity is equal to zero on equilibria, the "kinematic" component of the system of equations in variations, obtained from Eq. (4.11), has the form

$$\delta\dot{\boldsymbol{\theta}} = \delta\boldsymbol{\omega} \tag{4.12}$$

Equality (4.12), together with relations (2.5), enables us to represent Eqs. (4.7) in the form

$$\frac{d}{dt}\delta_\theta\left(\frac{\partial L}{\partial \boldsymbol{\omega}}\right) = \frac{\partial L}{\partial \boldsymbol{\omega}} \times \delta\dot{\boldsymbol{\theta}} + \delta_\theta\mathbf{Q}$$
$$\delta_\theta\left(\frac{\partial L}{\partial \boldsymbol{\omega}}\right) = \frac{\partial^2 L}{\partial \boldsymbol{\omega}^2}\delta\dot{\boldsymbol{\theta}} + \frac{\partial^2 L}{\partial \boldsymbol{\omega}\partial \boldsymbol{\alpha}}(\boldsymbol{\alpha} \times \delta\boldsymbol{\theta}) + \frac{\partial^2 L}{\partial \boldsymbol{\omega}\partial \boldsymbol{\beta}}(\boldsymbol{\beta} \times \delta\boldsymbol{\theta}) + \frac{\partial^2 L}{\partial \boldsymbol{\omega}\partial \boldsymbol{\gamma}}(\boldsymbol{\gamma} \times \delta\boldsymbol{\theta})$$
$$\delta_\theta\mathbf{Q} = \frac{\partial \mathbf{Q}}{\partial \boldsymbol{\omega}}\delta\dot{\boldsymbol{\theta}} + \frac{\partial \mathbf{Q}}{\partial \boldsymbol{\alpha}}(\boldsymbol{\alpha} \times \delta\boldsymbol{\theta}) + \frac{\partial \mathbf{Q}}{\partial \boldsymbol{\beta}}(\boldsymbol{\beta} \times \delta\boldsymbol{\theta}) + \frac{\partial \mathbf{Q}}{\partial \boldsymbol{\gamma}}(\boldsymbol{\gamma} \times \delta\boldsymbol{\theta}) \tag{4.13}$$

Hence, the problem now consists of investigating a system of three second-order equations.

Example. Continuation. The necessary conditions for the stability of the equilibria. In the light of the above discussion and relations (4.3) and (4.12) the "dynamic part" of the equations of perturbed motion

$$I_1\delta\dot{\omega}_1 = K_2\delta\omega_3 - K_3\delta\omega_2 + (I_3 - I_2)\left(\varkappa_\beta\delta\beta_3 + \varkappa_\gamma\delta\gamma_2\right) \quad (123, \alpha\beta\gamma)$$

can be written as

$$I_1\delta\ddot{\theta}_1 = K_2\delta\dot{\theta}_3 - K_3\delta\dot{\theta}_2 - k_1\delta\theta_1 \quad (123) \tag{4.14}$$

$$k_1 = (I_2 - I_3)(\varkappa_\beta - \varkappa_\gamma) \quad (123, \alpha\beta\gamma)$$

The characteristic equation of the system has the form

$$P(\mu) = I_1 I_2 I_3 \mu^3 + \left[\sum_{(123)} (I_1 K_1^2 + I_1 I_2 k_3)\right]\mu^2 + \left[\sum_{(123)} (I_1 k_2 k_3 + K_1^2 k_1)\right]\mu + k_1 k_2 k_3 =$$
$$= I_1 I_2 I_3 \left(\mu^3 + a\mu^2 + b\mu + c\right) = 0, \quad \mu = \lambda^2 \tag{4.15}$$

The necessary conditions for stability are satisfied if all three roots of Eq. (4.15) are real and negative.

As is well known from the general theory of polynomials, all three roots of Eq. (4.15) are real if its discriminant is positive:

$$a^2 b^2 - 4a^3 c + 18abc - 4b^3 - 27c^2 > 0$$

It can be shown,[15] that when this condition is satisfied all the roots of Eq. (4.15) are negative if and only if all its coefficients are positive.

Remark 5. This fact clarifies why one of the conditions obtained previously[16] on the basis of Hurwitz, criterion turned out to be surplus.

If $c > 0$ and the degree of instability is equal to zero, secular stability occurs. If $c < 0$, the degree of instability of the equilibrium is odd, and it is unstable by the instability theorem. We will consider the case when $c > 0$ and the degree of instability is equal two.

We will use the results obtained by Kozlov,[16] which not only offer a similar interpretation of the conditions for the roots of the cubic equation to be real and negative, but which also define the condition for the stabilizability of the equilibrium by a gyrostatic moment $|\mathbf{K}| \mapsto \infty$ that is of sufficiently large value. This condition has the following form in the notation employed

$$\Sigma = k_1 K_1^2 + k_2 K_2^2 + k_3 K_3^2 > 0 \tag{4.16}$$

Note that when condition (4.5) is satisfied the degree of instability is equal to two either when the following condition is satisfied

$$I_1 < I_3 < I_2$$

when the body is stretched along the field component with the least intensity and compressed along the field component with the mean intensity, or when the following condition is satisfied

$$I_2 < I_1 < I_3$$

when the body is stretched along the field component with the mean intensity and compressed along the field component with the greatest intensity.

Remark 6. The procedure described above also remains valid when describing equilibria in problems of the motion of rigid bodies or systems of rigid bodies with respect to rotating systems of coordinates, for example, orbital systems of rigid bodies or bodies suspended on a string, known for their unwieldy nature when described using angles. In these problems, depending on the formulation, either a reduced or a changed potential plays the role of the potential U.

Remark 7. Relations (1.6), (1.9), (2.6), (4.12) and also relation (4.11), written using a matrix commutator in the form

$$\frac{d}{dt}\delta\hat{\theta} = \delta\hat{\omega} + [\delta\hat{\theta}, \hat{\omega}]$$

also remain true for orthogonal matrices of arbitrary dimensions. They are used in problems of the motion of rigid bodies in spaces of higher dimensions.

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